Coherent Control of Tunneling in Double-Well Potentials

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(Dated: January 9, 2013)

Abstract

For an asymmetric double-well potential system, it is shown that, if the potential is quadratic until it reaches several times of the zero-point energies from the bottoms in each well, the energy eigenvalues of the low lying excited states of the double-well system must be close to the eigenvalues of the quadratic potentials. These eigenvalue structures suggest a method for the coherent control of the tunneling as well as realizing almost complete localization of the wave packet in one of the wells, by handling the double-well asymmetry. Numerical examples are included to indicate that the method could be useful also in a more general potential, and to propose experimental confirmations.

PACS numbers: 03.65.Xp, 03.65.Sq, 03.75.-b, 82.20.Xr

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I. INTRODUCTION

Quantum tunneling through a barrier is a fundamental physical effect [1, 2], and the neutral atoms trapped in optical lattices have given us an opportunity to study the aspects of tunneling, including coherent dynamics on macroscopic scale. For the symmetric double-well system which has long served as a paradigm of quantum physics, the eigenvalue structures are well-known for the low lying excited states [3–7], and it has been pointed out that, by adding a specific driving force, the tunneling dynamics can be brought to a complete standstill known as coherent destruction of tunneling (CDT) [7]. Recently this coherent destruction has been visualized in single particle tunneling [8] as well as in the tunneling of Bose-Einstein condensates (BECs) [9]. On the other hand, in experiments, localized wave packets have been prepared in one well of an asymmetric double-well potential, and the tunneling dynamics has been observed by turning off the asymmetric part of the potential [8, 10, 11]. It is also known that the density distributions of the BECs of interacting particles are asymmetric in the asymmetric double-well potentials [1, 12–14], to result in the non-vanishing relative phase evolution rate [15, 16].

In this article, we will find the energy eigenvalues of low lying excited states of an asymmetric double-well potential $V(x)$ which has one local maximum at $x = x_c$ between the wells, by constructing WKB wave functions with the quadratic connection formula. To quantify the degree of tunneling, we define the *tunneling visibility* for a wave function $\psi(x,t)$, as $\mathcal{V} = (P_{max} - P_{min})/(P_{max} + P_{min})$, where P_{max} (P_{min}) denotes the maximum (minimum) value of $P_r(t) = \int_{x_c}^{\infty} \psi^*(x,t)\psi(x,t)dx$ during the time evolution. We find that the wave functions of (almost) arbitrary $\mathcal{V}$ ranging from 0 to 1 can be realized from a Gaussian wave packet, by controlling the potential energy difference between the bottoms of the double well. The case of $\mathcal{V} \approx 0$ with $P_{max} \approx 1$ amounts to the CDT, and this case can be realized when the two wells can be considered to be separate. A Gaussian wave packet at a stand still can be realized also in the well of higher bottom.

While the results for a single particle can be applied only for the systems of noninteracting particles, we note that interacting systems have been extensively studied [17–20]. Particularly, for the systems of a few bosons, highly delayed pair tunneling analogous to nonlinear self-trapping has been found for the medium range of the interaction strength [21]. Though we only consider the double-well potentials bounded from below, the system of a particle

in a periodic potential with an additional constant force has been of great interest [22–24], and we note that this system has been analyzed through the instanton method [25] which is intimately related with the WKB analysis [3].

In the next section, before presenting the main *analytic* results, two asymmetric systems will be numerically studied. In Sec. III, we will construct the WKB wave functions for a general potential $V(x)$. It will be shown that the asymmetric systems can be classified into two different regimes: In the one regime, an eigenfunction of low lying excited states has significant amplitude in both wells as in the symmetric systems, while, in the other, the eigenfunction describes the particle mostly localized in just one of the wells. In Sec. IV, we will develop formulas for the estimation of the energy eigenvalues in the regime of the localized eigenfunctions, and for the estimation of the tunneling visibility in the other. In Sec. V, the asymmetric double oscillator model will be numerically solved to give an implication on the eigenvalue structure of a general double-well system, and to show that *WKB description* could be *remarkably accurate*. The last section will be devoted to a summary and discussions.

II. COHERENT CONTROL OF TUNNELING: NUMERICAL EXAMPLES

In this section, we will study two systems numerically to indicate that the coherent control method could also be useful in a general potential whose wells are *not* exactly quadratic, and to expose that gravity may be used to control the tunneling dynamics in the settings of the recent experiments [8, 10].

First, we consider the system of a particle of mass m in the quartic double-well potential

$$V_Q(\alpha; x) = \hbar\omega \left[-\frac{x^2}{4l_{ho}^2} + \frac{x^4}{96 l_{ho}^4} + \frac{\alpha x}{8\sqrt{3} l_{ho}} + C(\alpha) \right], \tag{1}$$

with $l_{ho} = \sqrt{\frac{\hbar}{m\omega}}$, where $C(\alpha)$ is introduced to ensure that the minimum of the potential is 0. While the term proportional to α is added to give the asymmetry, $V_Q(0; x)$ is a special case of the well-known potentials [3–7]. For $V_Q(0; x)$, the angular frequency for small oscillations at the bottoms of each well is ω, and the barrier height is $3\hbar\omega/2$.

For the Gaussian wave packet $\phi_G(\alpha; x) =< x|\phi_G(\alpha) >= \exp[-(x - a(\alpha))^2/(2l_r^2)]/(l_r^2\pi)^{1/4}$ centered at the bottom of the right well, $a(\alpha)$, with $l_r^{-4} = \frac{m^2\omega_r^2(\alpha)}{\hbar^2} = \frac{m}{\hbar^2}\frac{d^2V_\alpha(x)}{dx^2}|_{x=a(\alpha)}$, in Fig. 1, we evaluate the probabilities $P_i(\alpha) = | < \phi_G(\alpha)|\varphi_i(\alpha) > |^2$, where $\varphi_i(\alpha; x) =<$

$x|\varphi_i(\alpha)>$ are the eigenfunctions arranged in the order of ascending energy eigenvalue $E_i(\alpha)$ $(i = 0, 1, 2, \cdots)$. When we define the (unnormalized) wave function $\psi_\alpha(x,t)$ as

$$\begin{aligned}\psi_\alpha(x,t) & \\ = \quad & \sqrt{P_1(\alpha)}\varphi_1(\alpha;x)\exp\left[\frac{-iE_1(\alpha)t}{\hbar}\right] \\ & +\sqrt{P_2(\alpha)}\varphi_2(\alpha;x)\exp\left[\frac{-iE_2(\alpha)t}{\hbar}\right], \end{aligned} \tag{2}$$

Fig. 1 indicates that, in this system of deep quantum regime, $\psi_\alpha(x,0)$ closely describes the Gaussian wave function $\phi_G(\alpha;x)$ for $0.01 \leq \alpha$ in the given range of α. Further, plot (e) shows that, if $0.01 < \alpha < 0.9$, the Gaussian wave packet is closely described by an eigenfunction, $\varphi_1(\alpha;x)$, which is in turn mostly localized in the right well of the higher bottom (We note that localized eigenfunctions have also been known in the buried double-well systems [26]).

As is well-known in the systems of symmetric potentials [3, 5], the wave function $\frac{1}{\sqrt{2}}\left(\varphi_0(0;x)e^{\frac{-iE_0(0)t}{\hbar}} + \varphi_1(0;x)e^{\frac{-iE_1(0)t}{\hbar}}\right)$ describes the system which tunnels back and forth between an almost Gaussian state localized in the left well and the state localized in the right well, with $\mathcal{V} \approx 1$. As, for $0.01 \leq \alpha \leq 0.9$, the ground state wave function $\varphi_0(\alpha;x)$ and the first excited state wave function $\varphi_1(\alpha;x)$ are mostly localized in the left and right wells with the shapes close to a Gaussian, respectively, if we could change α from 0 to a value which is larger than 0.01 but smaller than 0.9, without distorting the wave function by the change, then we will have a system where the probability density is almost stationary. For this stationary system, the probability of finding the particle in the left (or right) well depends on the details of changing α. If the change could be made in a much shorter period of time compared with the period of tunneling $2\hbar\pi/(E_1(0) - E_0(0))$, the probability of finding the particle in one of the wells crucially depends on the timing of the change. If we tune α back to 0 or to 1, tunneling dynamics appears again, but in this time the tunneling visibility $\mathcal{V}$ could be less than 1 depending on the process.

Second, for an atomic spinor trapped in a double-well of the potential

$$\begin{aligned} V_L(\beta;x) \ = \ & CE_R[\cos^2 kx + \xi\cos^2(\frac{kx}{2}) - \xi/2 + \xi^2/16] \\ & +m\beta gx, \end{aligned} \tag{3}$$

with $E_R = (\hbar k)^2/2m$ [8, 10], we explicitly consider the case of $C = 10$ and $\xi = \frac{1}{2}$ which is similar to an experimental situation [8]. For $\beta = 0$, with the vanishing boundary condition

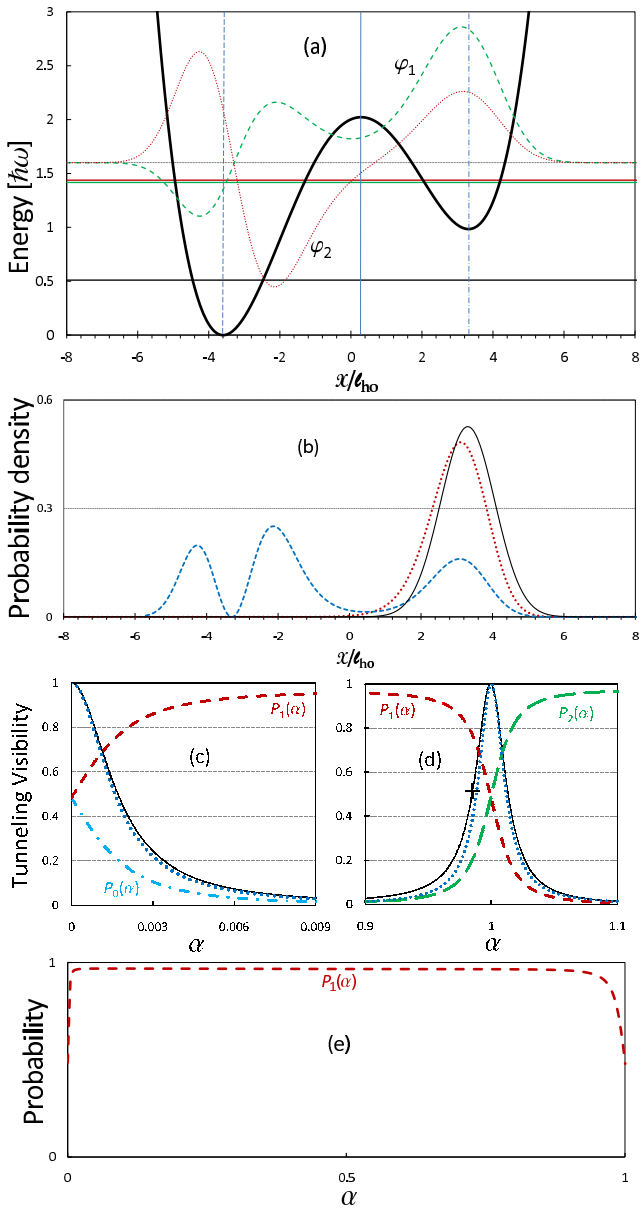

FIG. 1: (Color online) Tunneling dynamics in $V_Q(\alpha; x)$. (a) A potential (thick solid line) and three lowest eigenvalues (thin black, green, red solid lines), for $\alpha = \alpha_0 = 0.985$. The dotted and dashed, dashed, solid vertical lines indicate the values of $a(\alpha_0)$, $-b(\alpha_0)$, $x_c(\alpha_0)$, respectively, where $V_Q(\alpha_0; -b(\alpha_0)) = 0$. (b) $|\phi_G(\alpha_0; x)|^2$ (solid line), and $|\psi_{\alpha_0}(x,t)|^2$ at $t = 0$ (dotted line) and at $t = \hbar\pi/(E_2(\alpha_0) - E_1(\alpha_0))$ (dashed line). (c) and (d) The calculated tunneling visibility for $\psi_\alpha(x,t)$ (solid lines) and an estimation (dotted lines). The estimation is made from Eq. (16). For (c), $\delta_\epsilon/\delta_a = 2V_\alpha(a(\alpha))/(E_1(0) - E_0(0))$. For (d), since $E_2(\alpha) - E_1(\alpha)$ has the minimum at $\alpha = \alpha_1 = 1.00$, $2[V_\alpha(a(\alpha)) - V_{\alpha_1}(a(\alpha_1))]/(E_2(\alpha_1) - E_1(\alpha_1))$ is used as δ_ϵ/δ_a. The "+" mark denotes the value of α_0.

at $x = \frac{2j\pi}{k}$ (j :integer), there are three energy eigenvalues E_i^L ($i = 0, 1, 2$) under the barrier height, with $E_1^L - E_0^L = 0.122E_R$, $E_1^L + E_0^L = 5.70E_R$, and $E_2^L = 7.27E_R$.

If $\lambda = 2\pi/k = 811$ nm, $g = 980$ cm/s^2 and cesium atoms are in $V_L(1; x)$, the fact $mg\lambda/2 = 0.580E_R = 4.75(E_1^L - E_0^L)$ then suggests that the tunneling visibility is very low for a cesium atom trapped in the vertical optical lattice aligned along the Earth's gravity. If the right minimum of a double-well is located at x_r, we construct a Gaussian wave packet $\phi(f; x) =< x|\phi(f) >= \exp[-(x - x_r)^2/(2f^2 l_r^2)]/(f^2 l_r^2 \pi)^{1/4}$, here, with a fitting factor f. Indeed, for the first excited state $|\varphi_1^L >$ of $\beta = 1$, we find that $| < \phi(0.759)|\varphi_1^L > |^2 = 0.980$ and $| < \phi(1.00)|\varphi_1^L > |^2 = 0.945$, which shows that the eigenfunction $\varphi_1^L(x)$ is closely described by a Gaussian wave packet, to prove the low visibility of the wave packet. As the visibility is high for the symmetric horizonal lattice, this shows that gravity may be used to control the tunneling dynamics in the optical lattices.

III. WKB WAVE FUNCTIONS

In this section, we will construct WKB wave functions for a general potential $V(x)$, assuming that $V(x)$ is written as $\frac{m\omega_l^2}{2}(x+b)^2$ and as $\frac{m\omega_r^2}{2}(x-a)^2 + \epsilon\hbar\omega_r$ around the bottoms of the left and right wells, respectively. The eigenvalue structure will then be found by requiring the WKB wave functions to be asymptotically matched, in the overlapping regions, onto the exact solutions of the quadratic wells.

In the quadratic regions of $V(x)$, the eigenfunction is described by the parabolic cylinder function $D_\eta(z)$, and the eigenfunction of an energy eigenvalue $\hbar\omega_r(\nu + \epsilon + \frac{1}{2})[\equiv \hbar\omega_l(\mu + \frac{1}{2})]$ is written as

$$C_L D_\mu\left(-\frac{\sqrt{2}(x+b)}{l_l}\right) \text{ and } C_R D_\nu\left(\frac{\sqrt{2}(x-a)}{l_r}\right), \tag{4}$$

near the bottoms of the left and right wells, respectively, with $l_i = \sqrt{\frac{\hbar}{m\omega_i}}$ ($i = l, r$). On the other hand, by taking $x_c = 0$, in the region of the barrier we have an approximate solution for the eigenfunction through the WKB method [3–5], as

$$\begin{aligned}\psi_{WKB}(x) \;=\; & \frac{N_R\sqrt{\hbar}}{\sqrt{l_{ho}p(x)}}\exp\left[\int_0^x \frac{p(y)}{\hbar}dy\right] \\ & + \frac{N_L\sqrt{\hbar}}{\sqrt{l_{ho}p(x)}}\exp\left[-\int_0^x \frac{p(y)}{\hbar}dy\right],\end{aligned} \tag{5}$$

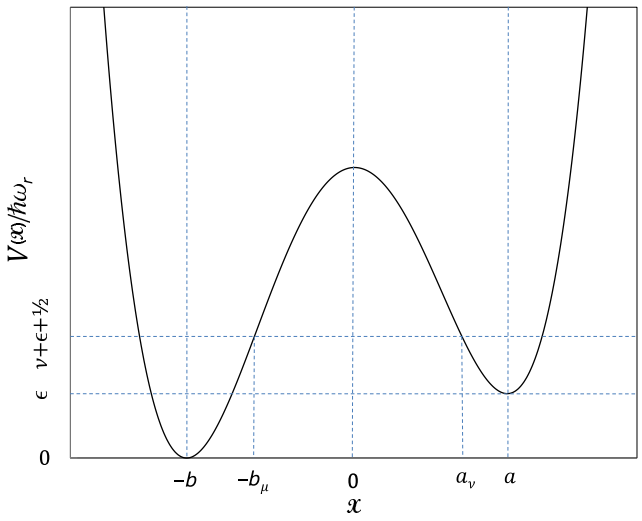

FIG. 2: (Color online) An illustration of turning points.

where $p(y) = \sqrt{2m[V(y) - (\nu + \epsilon + \frac{1}{2})\hbar\omega_r]}$. The relations between the real coefficients $C_L,\ C_R,\ N_L,\ N_R$ may be given by comparing the eigenfunctions in the regions where the descriptions by the parabolic cylinder function and by the WKB function are both valid. On the negative real axis, the asymptotic expansion of the parabolic cylinder function is [27]

$$\begin{aligned} D_\eta(z) \ \sim\ & \frac{\sqrt{2\pi}}{\Gamma(-\eta)} \exp\left[\frac{z^2}{4}\right] \frac{1}{|z|^{\eta+1}} \left[1 + O\left(\frac{\eta^2}{z^2}\right)\right] \\ & + \cos(\eta\pi) \exp[-\frac{z^2}{4}] |z|^\eta \left[1 + O\left(\frac{\eta^2}{z^2}\right)\right]. \end{aligned} \tag{6}$$

The WKB wave function can also be expanded, for instance, as

$$\begin{aligned} & \psi_{WKB}(x) \\ & \sim 2^{\frac{1}{4}} N_R \sqrt{\frac{l_r}{l_{ho}}} \left(\frac{e}{\nu + \frac{1}{2}}\right)^{\frac{\nu}{2}+\frac{1}{4}} \left(\frac{\sqrt{2}(a-x)}{l_r}\right)^\nu \\ & \qquad \times \exp\left(-\frac{(a-x)^2}{2l_r^2} + \int_0^{a_\nu} \frac{p(y)}{\hbar} dy\right) \\ & \quad + 2^{\frac{1}{4}} N_L \sqrt{\frac{l_r}{l_{ho}}} \left(\frac{\nu + \frac{1}{2}}{e}\right)^{\frac{\nu}{2}+\frac{1}{4}} \left(\frac{l_r}{\sqrt{2}(a-x)}\right)^{\nu+1} \\ & \qquad \times \exp\left(\frac{(a-x)^2}{2l_r^2} - \int_0^{a_\nu} \frac{p(y)}{\hbar} dy\right), \end{aligned} \tag{7}$$

when $\frac{a-x}{a-a_\nu} \gg 1$ and x is in the region of quadratic potential of the right well, where $a_\nu, -b_\mu$ are turning points satisfying $V(a_\nu) = V(-b_\mu) = (\nu + \epsilon + \frac{1}{2})\hbar\omega_r \quad (a > a_\nu > 0,\ b > b_\mu > 0)$ (see Fig. 2). By comparing the leading terms in the parabolic cylinder function description

which originates from the first term in the right hand side (r.h.s.) of Eq. (6) with the relevant terms in the WKB description, we have

$$
\begin{aligned}
N_R &= \frac{\sqrt{\sqrt{2}\pi}}{\Gamma(-\mu)}\sqrt{\frac{l_{ho}}{l_l}}\left(\frac{e}{\mu+\frac{1}{2}}\right)^{\frac{\mu}{2}+\frac{1}{4}} e^{\int_{-b_\mu}^{0}\frac{p(y)}{\hbar}dy}C_L, \\
N_L &= \frac{\sqrt{\sqrt{2}\pi}}{\Gamma(-\nu)}\sqrt{\frac{l_{ho}}{l_r}}\left(\frac{e}{\nu+\frac{1}{2}}\right)^{\frac{\nu}{2}+\frac{1}{4}} e^{\int_{0}^{a_\nu}\frac{p(y)}{\hbar}dy}C_R.
\end{aligned} \tag{8}
$$

If any of μ or ν is not close to a non-negative integer, in the large separation limit of $e^{\int_{-b_\mu}^{0}\frac{p(y)}{\hbar}dy} \gg 1$ and $e^{\int_{0}^{a_\nu}\frac{p(y)}{\hbar}dy} \gg 1$, Eq. (8) yields

$$
|N_R| \gg |C_L| \quad \text{and} \quad |N_L| \gg |C_R|. \tag{9}
$$

If the WKB condition $p^2(x) \gg \hbar|\frac{dp}{dx}|$ is satisfied, in the region of the barrier, the first term in r.h.s. of Eq. (5) with positive (negative) N_R is a monotonically increasing (decreasing) function and the second term with positive (negative)N_L is a monotonically decreasing (increasing) function. Eq. (9) then implies that if an eigenfunction exist for such ν, it gives the probability distribution in which the probability of finding the particle in the barrier region is considerable. Since $E_\varphi > \int \varphi^*(x)V(x)\varphi(x)$, if an eigenfunction gives a considerable probability in the barrier region, the eigenvalue can not be much smaller than $V(0)$.

On the other hand, if any of ν or μ is close to a non-negative integer, the relations in Eq. (9) are not valid. If ν is close to a non-negative integer, due to the singularity in the gamma function, the second term of the r.h.s. of Eq. (6) can also be a leading term. By comparing this type of leading term in the parabolic cylinder function description with the relevant term in the WKB wave function, we have a relation between N_R and C_R. By combining this relation with the first one in Eq. (8), we have

$$
\begin{aligned}
\frac{C_L}{C_R} &= \sqrt{\frac{l_l}{l_r}}\frac{\cos(\nu\pi)\Gamma(-\mu)}{\sqrt{2\pi}}\left(\frac{\nu+\frac{1}{2}}{e}\right)^{\frac{\nu}{2}+\frac{1}{4}} \\
&\quad \times \left(\frac{\mu+\frac{1}{2}}{e}\right)^{\frac{\mu}{2}+\frac{1}{4}} \exp\left[-\int_{-b_\mu}^{a_\nu}\frac{p(y)}{\hbar}dy\right].
\end{aligned} \tag{10}
$$

If μ is not close to an integer, Eqs. (8) and (10) imply that the eigenfunction gives the probability distribution in which the particle is mostly found in the right well.

If μ is close to a non-negative integer, we have the relation

$$
\frac{C_L}{C_R} = \sqrt{\frac{l_l}{l_r}}\frac{\sqrt{2\pi}}{\cos(\mu\pi)\Gamma(-\nu)}\left(\frac{e}{\nu+\frac{1}{2}}\right)^{\frac{\nu}{2}+\frac{1}{4}}
$$

$$\times \left(\frac{e}{\mu+\frac{1}{2}}\right)^{\frac{\mu}{2}+\frac{1}{4}} \exp\left[\int_{-b_\mu}^{a_\nu} \frac{p(y)}{\hbar} dy\right]. \tag{11}$$

In this case, if ν is not close to an integer, Eqs. (8) and (11) imply that the eigenfunction gives the probability distribution of the particle mostly localized in the left well.

For an eigenstate whose eigenvalue is much lower than the barrier height, the eigenvalue thus must be close to $\hbar\omega_l(m+\frac{1}{2})$ or $\hbar\omega_r(n+\frac{1}{2}+\epsilon)$ $(n, m = 0, 1, 2, \cdots)$, the eigenvalues of the quadratic potentials of the wells. Furthermore, the fact $D_k(\sqrt{2}y) = e^{-y^2/2}H_k(y)/(\sqrt{2})^k$ for a non-negative integer k, implies that, if the eigenvalue is close to $\hbar\omega_r(n+\frac{1}{2}+\epsilon)$ $(\hbar\omega_l(m+\frac{1}{2}))$, the eigenfunction of the double-well system must be closely described by $\psi_n^{ho}(a; l_r; x)$ $(\psi_m^{ho}(-b; l_l; x))$ around the bottom of the right (left) well, with an eigenfunction of a simple harmonic oscillator $\psi_k^{ho}(c; l; x)$ $(\equiv H_k(\frac{x-c}{l})\exp[-\frac{(x-c)^2}{2l^2}]/\sqrt{\sqrt{\pi}l2^k k!})$.

IV. TWO DIFFERENT REGIMES

The analysis of the previous section shows that an eigenfunction of the low lying excited states in the large separation limit has significant amplitude either in both wells or in just one of the wells. In this section, we will show that the eigenfunction of significant amplitude in both wells must be accompanied by another eigenfunction to form a doublet. As in the symmetric case, a linear combination of the doublet is responsible for the tunneling, and we will calculate the tunneling visibility for the combination. For the eigenfunctions localized in one of the wells, we will develop a formula for the energy eigenvalue estimation.

A. Tunneling visibility

For the case that both μ and ν are close to integers m and n, respectively, we define $\delta_\epsilon = \epsilon + n - \frac{\omega_l}{\omega_r}m$, so that $|\delta_\epsilon|$ is equal or less than the minimum of $\frac{1}{2}$ and $\frac{\omega_l}{2\omega_r}$. In this case, the corresponding eigenfunction gives considerable probabilities in both of the left and right wells, and μ and ν should be written as $\mu = m + \delta_\mu$ and $\nu = n + \delta_\nu$ with $|\delta_\mu|$, $|\delta_\nu|$, $|\delta_\epsilon| \ll 1$. From the fact that $\delta_\mu = \frac{\omega_r}{\omega_l}(\delta_\nu + \delta_\epsilon)$, Eqs. (10) and (11) then yield $\delta_\nu^2 + \delta_\epsilon\delta_\nu - \delta_a^2 = 0$, with

$$\begin{aligned}\delta_a \;=\; & \sqrt{\frac{\omega_l}{\omega_r}}\sqrt{\frac{1}{2\pi n!m!}}\left(\frac{n+\frac{1}{2}}{e}\right)^{\frac{2n+1}{4}} \\ & \times \left(\frac{m+\frac{1}{2}}{e}\right)^{\frac{2m+1}{4}} \exp\left[-\int_{-b_m}^{a_n} \frac{p(y)}{\hbar} dy\right].\end{aligned} \tag{12}$$

With

$$\delta_{\pm} = \frac{1}{2}(-\delta_{\epsilon} \pm \sqrt{\delta_{\epsilon}^2 + 4\delta_a^2}), \tag{13}$$

when $\delta_\nu = \delta_-$, the eigenfunction is written as

$$\psi^-(x) = \frac{(-1)^n \delta_a \psi_m^{ho}(-b; l_l; x) + \delta_+ \psi_n^{ho}(a; l_r; x)}{\sqrt{\delta_a^2 + \delta_+^2}}, \tag{14}$$

while the eigenfunction of $\delta_\nu = \delta_+$ is

$$\psi^+(x) = \frac{-(-1)^n \delta_+ \psi_m^{ho}(-b; l_l; x) + \delta_a \psi_n^{ho}(a; l_r; x)}{\sqrt{\delta_a^2 + \delta_+^2}}. \tag{15}$$

The formal expression of $\delta_\pm$ in terms of δ_ϵ and δ_a can be understood from the fact that, for $|\delta_\epsilon| \ll 1$, the tunneling dynamics is essentially described by that of a two-level system(TLS) [7, 30, 31]. If $\psi_n^{ho}(a; l_r; x)$ is written as a linear combination of $\psi^-(x)$ and $\psi^+(x)$, the visibility of the linear combination is

$$\mathcal{V} = 1/[1 + \frac{1}{2}(\frac{\delta_\epsilon}{\delta_a})^2]. \tag{16}$$

For the visibility estimation in Fig. 1(c) and (d), the parameter is determined from the consideration that, when $\delta_\epsilon = 0$, the energy splitting is given as $2\hbar\omega_r\delta_a$. The fact that the tunneling dynamics significantly takes place for $|\delta_\epsilon| \ll 1$, even with $m \neq n$, may be closely related to the resonant enhancement of tunneling in the multiple-well structures [22, 26]. As in the BEC loaded into an asymmetric double-well potential [12], if there are noninteracting N atoms in the ground state of $V(x)$ of small δ_ϵ ($\ll \delta_a$), the number difference between the left and right wells is proportional to δ_ϵ.

B. Energy eigenvalue estimation

For an eigenfunction $\psi(x)$ of the system of $V(x)$ with the energy eigenvalue E, we have the identity

$$\begin{aligned} & E - \hbar\omega(n + \frac{1}{2}) \\ & = \frac{\int_{-\infty}^{\infty} \left[V(x) - V_r^{ho}(x)\right] \psi(x) \psi_n^{ho}(a; l_r; x) dx}{\int_{-\infty}^{\infty} \psi(x) \psi_n^{ho}(a; l_r; x) dx}, \end{aligned} \tag{17}$$

where $V_r^{ho}(x) = \frac{\hbar\omega_r}{2}\frac{(x-a)^2}{l_r^2}$. In numerical calculations, this identity may be efficiently used in estimating the energy eigenvalue of an eigenfunction which is close to $\psi_n^{ho}(a; l_r; x)$.

As the visibility also implies, when δ_ϵ is much larger than δ_0, $\psi(x)$ of $E \approx \hbar w_r(n+\epsilon+\frac{1}{2})$ is mostly localized in the right well, and around the bottom it will be closely described by $\psi_n^{ho}(a; l_r; x)$, to give $\psi_n^{app}(l_r; x)$ an approximation of $\psi(x)$ in this well. In the other regions, Eqs. (5,7) and the WKB method can be used to find $\psi_n^{app}(l_r; x)$. Eq. (17) then may be used to find a correction to ν, as

$$\begin{aligned} &\nu - n + \epsilon \\ &\approx \frac{\int_{-\infty}^{\infty}[V(x) - V_r^{ho}(x)]\psi_n^{app}(l_r; x)\psi_n^{ho}(a; l_r; x)dx}{\hbar\omega_r \int_{-\infty}^{\infty} \psi_n^{app}(x)\psi_n^{ho}(a; x)dx}. \end{aligned} \tag{18}$$

This approximation of a localized eigenfunction (ALE) can also be made similarly, for the eigenstate of $E \approx \hbar w_l(m+\frac{1}{2})$ which describes a probability distribution mostly localized in the left well.

V. PRECISION TEST: ASYMMETRIC DOUBLE OSCILLATOR

In application of the WKB method for a symmetric double-well potential, it is known that the energy splitting could be found accurately, if the (ground state) energy eigenvalue and thus the turning points are appropriately chosen [4]. If a well is quadratic with angular frequency ω, then $(j+\frac{1}{2})\hbar\omega$ (j: nonnegative integer) may be a good estimation for an energy eigenvalue.

In order to check the accuracy of the formalism we have provided, avoiding the turning-point problem as much as possible, we consider the system of the asymmetric double oscillator potential [28]

$$V_D(\epsilon; x) = \begin{cases} \hbar\omega \left(\frac{x + \sqrt{a^2 + 2\epsilon l_{\hbar 0}^2}}{\sqrt{2} l_{ho}} \right)^2 & \text{for } x < 0, \\ \hbar\omega \left[\left(\frac{x-a}{\sqrt{2} l_{ho}} \right)^2 + \epsilon \right] & \text{for } x \geq 0. \end{cases} \tag{19}$$

For this system, since both wells are exactly quadratic, the eigenfunctions are described by the parabolic cylinder functions on both sides of $x = 0$ [28, 29] and the continuities of the eigenfunction and its derivative at $x = 0$ can be used to find the eigenvalues $E_0^D(a), E_1^D(a), \cdots$. As in Fig. 3, the calculations indeed show that, when a is a few times of l_{ho}, the eigenvalues of the low lying excited states are close to $\hbar\omega(n+\epsilon+\frac{1}{2})$ or $\hbar\omega(m+\frac{1}{2})$. To expose that the estimation through the ALE is not valid when $\hbar\omega\delta_\epsilon$ is order

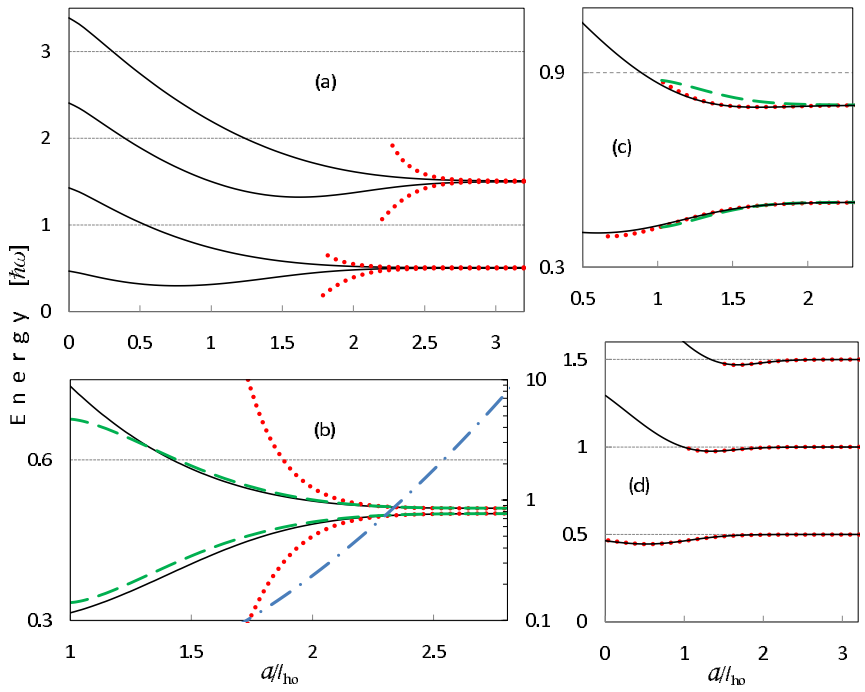

FIG. 3: (Color online) Energy eigenvalues for some lowest eigenstates of the system of $V_D(\epsilon; x)$. ϵ is chosen as 0.01 for (a) and (b), 0.3 for (c), and 0.5 for (d). Calculated values (Solid lines); estimated values through the ALEs (dotted lines) and through the approximation to a TLS (dashed lines). For the ALEs of the ground and second excited states, Eq. (11) and the turning points a_n, b_n satisfying $V_D(\epsilon; a_n) = V_D(\epsilon; -b_n) = (n + \frac{1}{2})\hbar\omega$ are used with $n = 0$ and $n = 1$, respectively; and for the first and third excited states, Eq. (10) and a_n, b_n of $V_D(\epsilon; a_n) = V_D(\epsilon; -b_n) = (n + \epsilon + \frac{1}{2})\hbar\omega$ are used with $n = 0$ and $n = 1$, respectively. In the approximation to a TLS, a_0 and b_0 are determined from $V_D(0; a_0) = V_D(0; -b_0) = \frac{1}{2}\hbar\omega$.

of or smaller than the energy difference of the adjacent energy eigenstates, we add the ratio $2\epsilon\hbar\omega/(E_1^D(a) - E_0^D(a))$ (dotted and dashed line) in Fig. 3(b).

When ϵ is as large as 0.3, the ALE gives better results than the approximation to a TLS practically in the whole range where both methods are applicable [Fig. 3(c)]. As the approximation to a TLS is suggested by the WKB method, Fig. 3 indeed shows that *WKB description* could be *very accurate*. Fig. 1(c) and (d) also suggests that this accuracy is not limited to the systems of the wells which are exactly quadratic. This with the reasons the WKB method provides implies that, if the potential $V(x)$ is quadratic until it reaches several times of the zero-point energies $\hbar\omega_l/2$ and $\hbar\omega_r/2$ from the bottoms of the left and right well, respectively, the energy eigenvalues of the low lying excited states of the system

must be close to the eigenvalues of the quadratic potentials.

VI. CONCLUSIONS AND OUTLOOK

We have shown, through the WKB method of quadratic connection formula, that the systems of asymmetric double-well potentials can be classified into two different regimes. In the regime of eigenfunctions giving significant amplitude in both wells, the tunneling dynamics could take place, while there is no tunneling in the regime of localized eigenfunctions. In this respect, the systems of the eigenfunctions mostly localized in just one of the wells are very different from those of the symmetric potentials. As Fig. 1(c), (d) and Fig. 3 clearly show, the WKB description could be very accurate, and the results given here may be valid for a system of the potential whose wells are not exactly quadratic.

For the regime of localized eigenfunctions, even in the deep quantum limit, it may be possible to confine a large number of noninteracting bosons in just one of the wells. For single-component fermions, in the light of the particle density $\rho_R(\epsilon; x, t) = \sum_i \psi^*_{Ri}(\epsilon; x, t)\psi_{Ri}(\epsilon; x, t)$ (see, e.g., Ref. [32]), the number of fermions which can be confined in one of the wells is limited by that of the localized eigenfunctions $\psi_{Ri}(\epsilon; x, t)$. For the system of particles confined in just one of the wells, the tunneling dynamics can be initiated and controlled by adjusting ϵ the potential energy difference between the bottoms of the double well, since, if we change ϵ so that $\delta_\epsilon \ll 1$, $\psi_{Ri}(\epsilon; x, t)$ turns into a linear combination of the eigenfunctions of the new system.

In the periodic arrangement of double-wells of the optical lattice where the tunneling is accompanied by a precession of the atom's angular momentum [8, 10], a considerable time-periodic *fluctuation of the population* of atoms in a spin state could imply that the atomic spinors are in the states of *high visibility*. Since δ_a is very small in the large separation limit and the period of tunneling is inversely proportional to δ_a, if a tunneling phenomenon can be established over a long period of time, it can be used for precision measurements.

Acknowledgments

The author thanks Professors Kyungwon An and Yong-il Shin for discussions on experimental aspects.

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